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Quasiperiodic patterns generated by mixing lattices derived from a dodecahedral star and an icosahedral star

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Abstract

A 3D quasiperiodic pattern by projection from an nD lattice can be defined by an orthonormal $n \times n$ *lattice matrix* which produces basis vectors in pattern space with a prescribed arrangement and basis vectors in perpendicular or *test space* satisfying the quasicrystallographic condition. A 16×16 *lattice matrix* is derived which produces basis vectors in pattern space as a combination or *mixing* of dually positioned dodecahedral star and icosahedral star. It is shown that the mixed star constitutes a eutactic star. Since the module generated by the mixed eutactic star is totally irrational, patterns generated by projection using the *lattice matrix* are quasicrystallographic and the equilateral pattern is generated for 1 : 1 mixing.

1. Introduction

In a previous paper (Soma & Watanabe, 1992), modifications of Beenker's pattern were discussed in terms of rotation of pattern and test space with respect to 4D lattice space. It is pointed out that Beenker's pattern can be regarded as a combination or a mixing of two square lattices of equal length with $\pi/4$ rotational separation. One of the modifications discussed is a change in size of two square lattices, keeping the rotational separation fixed as $\pi/4$. Beenker's pattern is a special case of this with 1:1 mixing. It is shown that by changing the mixing ratio both the periodic and *quasicrystallographic* patterns are generated. Similar treatment is performed by Duneau (1991). It should be pointed out that the mixing is not restricted to two lattices and it can also be extended to the 3D case (Soma & Watanabe, 1997, 1998). Baake *et al.* (1991, 1993) formulate the mixing as a Schul rotation and treat both 2D and 3D cases.

In this paper, the mixing of two lattices is discussed in §2; in §3, *lattice matrices* for lattices derived from a dodecahedral star and an icosahedral star are given in the form discussed in §4; and, in §4, the mixing of these lattices is discussed giving a 16×16 *lattice matrix*.

2. Mixing of two lattices

A 1D pattern or tiling by cut-and-project method (Katz & Duneau, 1986) from a 2D lattice is defined by specifying two 1×2 projection matrices; one specifying the projection from the lattice to pattern space defining the basis vectors and a *unit pattern* or the projection of a unit square to pattern space; the other specifying the projection from the lattice to test space defining the basis vectors and a *test window* or the projection of a unit square to test space. Combining these two matrices row-wise, we obtain

$$A_2(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad (1)$$

where θ is the angle between the x axis and the 1D pattern axis (space). This matrix is an orthonormal matrix on 2D space. Since its columns form the basis of a lattice that projects to the corresponding modules in pattern and test space, we call it a *lattice matrix*. It should be pointed out that the modification of basis vectors in pattern space can be realized as a rotation of pattern and test space with respect to the original lattice.

It is interesting to note that this 1D pattern by two prototiles can be regarded as a mixing of two 1D lattices of different sizes. Rewriting (1) as

$$A_2(r) = \frac{1}{(1+r^2)^{1/2}} \begin{pmatrix} 1 & r \\ -r & 1 \end{pmatrix}, \qquad (2)$$

where $r = \tan\theta$, the pattern generated by this lattice matrix is a mixing of 1D lattices of sizes 1 and r, where r is the *mixing ratio*, the ratio of the lengths of the basis vectors of two lattices. Quasicrystallographic patterns are generated if the basis vectors in test space are linearly independent over the rational number, *i.e.* r is irrational, and periodic or crystallographic patterns if r is rational.

It should be pointed out that the first row vector in (2) is a special case of more general row vector of a k 1D eutactic star as

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$$\frac{1}{\left(1+\sum_{i=2}^{k} r_i^2\right)^{1/2}}\begin{pmatrix}1 & r_2 & r_3 & \dots & r_k\end{pmatrix}, \tag{3}$$

where r_i is the length of stars with the first one as a unit or the mixing ratio of respective stars with respect to the first one, and they can be lifted to an orthogonal frame of kD space (Senechal, 1995), that is, the existence of *lattice matrix* for projection from kD space is guaranteed. For a k nD eutactic star, the mixed star is given by a matrix B with mixing parameters δ_i as

$$B = \begin{pmatrix}B_1 & \delta_2 B_2 & \delta_3 B_3 & \dots & \delta_k B_k\end{pmatrix}, \tag{4}$$

where B_i is an $n \times m_i$ matrix whose column vectors represent an nD $2m_i$ star. Assuming

$$B_i B_i^T = I_n, \tag{5}$$

where I_n is the $n \times n$ identity matrix and B_i^T is the transpose of B_i, we have

$$BB^T = \left(1 + \sum_{i=2}^{k} \delta_i^2\right) I_n, \tag{6}$$

which shows that the mixed star is also a eutactic star and they can be lifted to an orthogonal frame of mD space where $m = \sum_{i=1}^{k} m_i$.

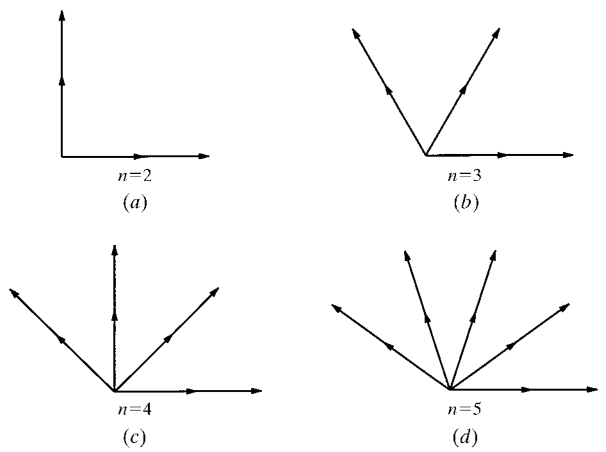

Fig. 1. Examples of 2D mixing by allocating two 1D vectors to each of the successive vectors of a $2n$-star.

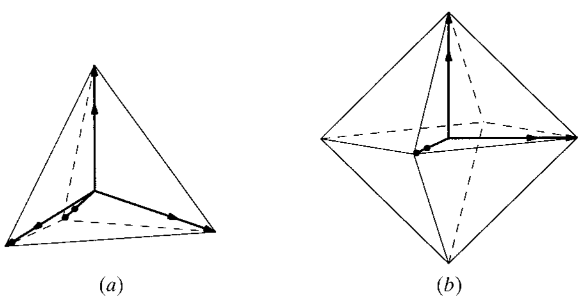

Fig. 2. Examples of 3D mixing by allocating two 1D vectors to vectors of regular polyhedral stars.

The simplest example for 2D mixing of 1D patterns of two prototiles is to take a Cartesian product of two 1D patterns allocating a 1D pattern to each vector of a star of angle $\pi/2$ (Fig. 1*a*) (Pleasants, 1985). It is a mixing of two square lattices of different sizes. The pattern consists of three prototiles, two squares and a rectangle. Generally, allocate the 1D pattern to each vector of a star of angle π/n, $n = 2, 3, \dots$ (Fig. 1). The case for $n = 3$ is reported by Warrington *et al.* (1998), using a dual of the Fibonacci-related Amman bar tri-grid formulation. For 3D, allocate the 1D pattern to each vector of tetrahedral, octahedral (Fig. 2), dodecahedral and icosahedral stars. The cases in Fig. 2 are discussed by Soma & Watanabe (1998).

Another example of mixing is to consider a $2n$ star with angle π/n and divide vectors into two groups selecting every other vector for each group, and mix the two groups by multiplying by a factor δ for one group (Fig. 3). The case of an 8-star is considered by Baake *et al.* (1991, 1993), Duneau (1991) and Soma & Watanabe (1992). This case is regarded as a mixing of two embedding 4-stars out of the 8-star. For 3D, consider the star of a dual pair of regular polyhedral stars with a factor δ for one of the pair (Fig. 4); the pair of hexahedral star and octahedral star is considered by Soma & Watanabe (1997), and the pair of dodecahedral star and icosahedral star is reported by Soma & Watanabe (1996) and discussed in the following sections. The mixing of embedding hexahedral stars of the dodecahedral star is discussed by Baake *et al.* (1991, 1993).

3. Lattice matrices for a dodecahedral star and an icosahedral star

A lattice matrix producing ten vectors from the center to the vertices of a regular pentagonal dodecahedron in pattern space described in Kramer & Haase (1989) and

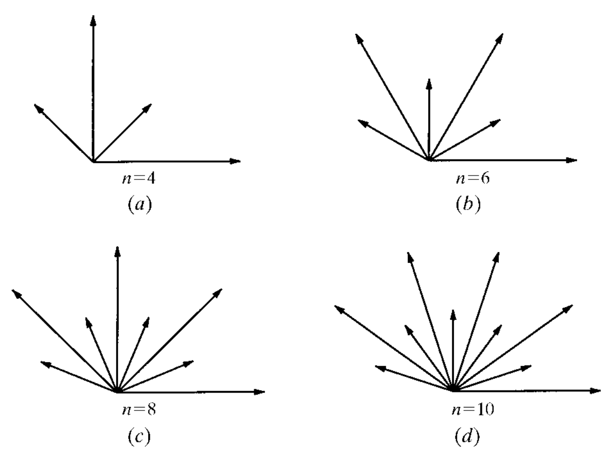

Fig. 3. Examples of 2D mixing by allocating a 1D vector to successive n vectors of a $2n$-star with a mixing parameter multiplying every other vector.

Watanabe (1994) adopts the coordinate system such that the coordinate (x, y, z) of one of the vertices of the embedded cube is $(1, 1, 1)$. For the discussion in the following section, the matrix is reproduced here with the change of coordinates as

$$A_{10} = \left(\frac{2}{5(1+\tau^2)}\right)^{1/2} \begin{pmatrix} S & \tau S \\ (\tau+1)H & (\tau-1)H \\ S' & \tau S' \\ \tau S & -S \\ \tau S' & -S' \\ (\tau-1)H & -(\tau+1)H \end{pmatrix}, \quad (7)$$

where

$$S = \begin{pmatrix} 1 & \cos\alpha & \cos 2\alpha & \cos 3\alpha & \cos 4\alpha \\ 0 & \sin\alpha & \sin 2\alpha & \sin 3\alpha & \sin 4\alpha \end{pmatrix},$$

$$S' = \begin{pmatrix} 1 & \cos 2\alpha & \cos 4\alpha & \cos\alpha & \cos 3\alpha \\ 0 & \sin 2\alpha & \sin 4\alpha & \sin\alpha & \sin 3\alpha \end{pmatrix},$$

$$H = \begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 & 1/2 \end{pmatrix},$$

$\tau = (1 + 5^{1/2})/2$ and $\alpha = 2\pi/5$. The z axis is coincident with one of the fivefold axes and the x axis is parallel to the line passing one of the vertices of a pentagonal face from its center. It can be shown that the 3D column vectors by the first three rows constitute a eutactic star (Fig. 4*b*).

The lattice matrix producing six vectors from the center to the vertices of an icosahedron in pattern space is given by Katz & Duneau (1986) and is reproduced here as

$$A_6 = (2/5)^{1/2} \begin{pmatrix} S & Z \\ H & 5^{1/2}/2 \\ S' & Z \\ H & -5^{1/2}/2 \end{pmatrix}, \quad (8)$$

where $Z = (0\,0)^T$, and the z axis is coincident with one of the vectors and the x axis is in a plane defined by the z-axis vector and one of its nearest vectors. As in the previous case, it can be shown that the 3D column vectors by the first three rows constitute a eutactic star (Fig. 4*b*).

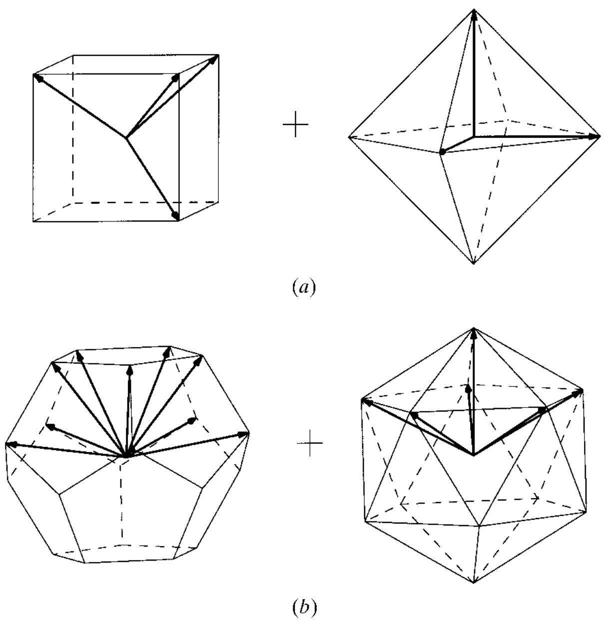

Fig. 4. Examples of 3D mixing by a dual pair of regular polyhedral stars.

4. Mixing a dodecahedral star and an icosahedral star

Consider 16 basis vectors in pattern space by mixing a dodecahedral star of the first three rows in (7) and an icosahedral star of the first three rows in (8). The icosahedral star in (8) should be rotated through π before mixing so that the two polyhedrons are in dual position. Introducing the mixing parameter δ, a 16×16 lattice matrix is obtained as

$$A_{16}(\delta) = \left(\frac{2}{5(1+\tau^2+\delta^2)}\right)^{1/2} \times \begin{pmatrix} S & \tau S & -\delta S & Z \\ (\tau+1)H & (\tau-1)H & \delta H & 5^{1/2}\delta/2 \\ S' & \tau S' & -\delta S' & Z \\ \tau S & (\delta^2 D - 1)S & \tau\delta D S & Z \\ \tau S' & (\delta^2 D - 1)S' & \tau\delta D S' & Z \\ (\tau-1)H & -(\tau+1)H & \delta H & -5^{1/2}\delta/2 \\ -\delta S & \tau\delta D S & (\tau^2 D - 1)S & Z \\ -\delta S' & \tau\delta D S' & (\tau^2 D - 1)S' & Z \\ \delta H & \delta H & -(\tau-1)H & -5^{1/2}(\tau+1)/2 \\ \delta H & -\delta H & -(\tau+1)H & 5^{1/2}(\tau-1)/2 \end{pmatrix}, \quad (9)$$

where D is the root of a quadratic equation $(\tau^2 + \delta^2)x^2 - 2x - 1 = 0$.

It is shown that the *unit pattern* or the 16D unit cube projected to pattern space is an enneacontahedron truncated by a triacontahedron. The depth of truncation varies as δ, which can take any value $[0, \infty]$; an enneacontahedron is obtained when δ is 0 and a triacontahedron when δ is infinity. The mixing ratio, the ratio of the length of a dodecahedral and an icosahedral vector, is $1 : 5^{1/2}\delta/[3(\tau+2)]^{1/2}$ and, for $\delta = [3(\tau+2)/5]^{1/2}$ or $1:1$ mixing, an equilateral truncated rhombic enneacontahedron (Watanabe & Betsumiya, 1992) is obtained (Fig. 5). For $\delta = 5(\tau+1)/[3(\tau+2)]^{1/2}$, the icosahedral star is in the lattice generated by the dodecahedral star and *vice versa* for $\delta = (\tau+2)^{1/2}/3^{1/2}(\tau+1)$. Since the module generated by each star is totally irrational, patterns generated by projection are quasicrystallographic for any values of δ. By projection from a 16D lattice using a lattice matrix with $1:1$ mixing, an equilateral 3D quasicrystallographic pattern is obtained

(Fig. 6) with 16 different prototiles (five from the combinations of dodecahedral vectors, two from those of icosahedral vectors and nine from those of mixing of the two). It is found that the 13D test polytope (the projection of the 16D unit cube to test space) has 2628 faces.

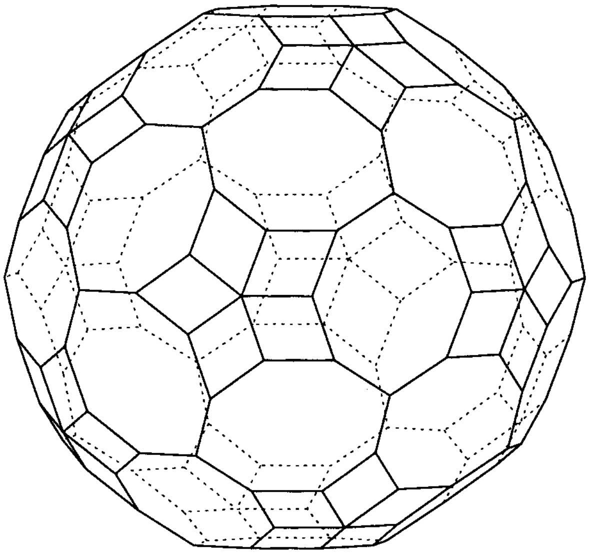

Fig. 5. A projection of a 16D unit cube to pattern space by the lattice matrix (9) with $\delta = [3(\tau + 2)/5]^{1/2}$ or 1 : 1 mixing; an equilateral truncated rhombic enneacontahedron.

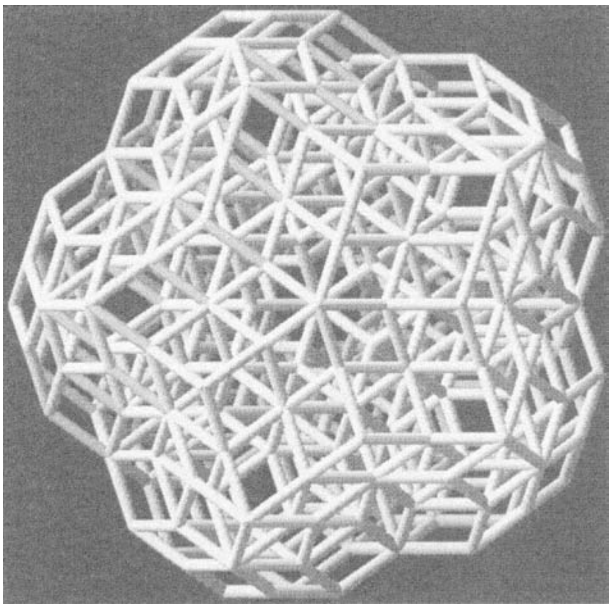

Fig. 6. A 3D quasicrystallographic pattern or tiling generated by the lattice matrix (9) with $\delta = [3(\tau + 2)/5]^{1/2}$ and $D = [1 + (1 + \tau^2 + \delta^2)^{1/2}]/(\tau^2 + \delta^2)$. The pattern is a part which is generated by the projection of points $\{(x_i)|x_i = \{0, 1\}\}$ in 16D lattice space. The test polytope is positioned so that its center is at the origin. The pattern consists of 16 different prototiles.

5. Concluding remarks

The concept of mixing two lattices for generating a new pattern is utilized to generate 3D quasiperiodic patterns by mixing lattices derived from a pentagonal dodecahedral star and an icosahedral star. It is interesting to apply this concept to other cases or to extend the concept to higher dimensions. Some of the examples given in §2 having icosahedral symmetry could be a model for clusters in quasicrystals having icosahedral symmetry.

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